

ON THE DEGENERATE FROBENIUS-EULER POLYNOMIALS

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ABSTRACT. In this paper, we consider the degenerate Frobenius-Euler polynomials and investigate some identities of those polynomials.

1. INTRODUCTION

For $u \in \mathbb{C}$ with $u \neq 1$, the *Frobenius-Euler polynomials* are defined by the generating function to be

$$\frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} H_n(x|u) \frac{t^n}{n!}, \quad (\text{see [1-10]}). \quad (1.1)$$

When $x = 0$, $H_n(x|u) = H_n(0|u)$ are called the *Frobenius-Euler numbers*.

From (1.1), we have

$$H_n(x|u) = \sum_{l=0}^n \binom{n}{l} H_l(u) x^{n-l}, \quad (n \geq 0), \quad (\text{see [6-8]}). \quad (1.2)$$

Note that

$$\frac{d}{dx} H_n(x|u) = n H_{n-1}(x|u), \quad (n \in \mathbb{N}).$$

In [3], L. Carlitz define the degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x|\lambda) \frac{t^n}{n!}. \quad (1.3)$$

When $x = 0$, $\beta_n(\lambda) = \beta_n(0|\lambda)$ are called the *degenerate Bernoulli numbers*. From (1.3), we have

$$\lim_{\lambda \rightarrow 0} \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (1.4)$$

where $B_n(x)$ are called *Bernoulli polynomials*

By (1.3) and (1.4), we get

$$\lim_{\lambda \rightarrow 0} \beta_n(x|\lambda) = B_n(x), \quad (n \geq 0).$$

The *Stirling number of the first kind* is defined as

$$(x)_n = \sum_{l=0}^n S_1(n, l) x^l, \quad (n \geq 0), \quad (1.5)$$

where $(x)_n = x(x-1) \cdots (x-n+1)$, $(x)_0 = 1$.

By (1.3), we easily get

$$\beta_n(x|\lambda) = \sum_{l=0}^n \binom{n}{l} \beta_{n-l}(x|\lambda)(x|\lambda)_l, \quad (n \geq 0), \quad (1.6)$$

where $(x|\lambda)_n = x(x-\lambda) \cdots (x-(n-1)\lambda)$.

The *Stirling numbers of the second kind* is defined by

$$x^n = \sum_{l=0}^n S_2(n, l)(x)_l, \quad (n \geq 0). \quad (1.7)$$

In this paper, we consider the degenerate Frobenius-Euler polynomials and investigate some properties and identities of those polynomials.

2. DEGENERATE FROBENIUS-EULER POLYNOMIALS

For $u \in \mathbb{C}$ with $u \neq 1$, we consider degenerate Frobenius-Euler polynomials which are given by the generating function to be

$$\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} h_{n,\lambda}(x|u) \frac{t^n}{n!}. \quad (2.1)$$

When $x=0$, $h_{n,\lambda}(u) = h_{n,\lambda}(0|u)$ are called *degenerate Frobenius-Euler numbers*.

From (2.1), we have

$$\begin{aligned} \frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u} (1+\lambda t)^{\frac{x}{\lambda}} &= \left(\sum_{l=0}^{\infty} \frac{h_{l,\lambda}(u)}{l!} t^l \right) \left(\sum_{m=0}^{\infty} (x|\lambda)_m \frac{t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} h_{l,\lambda}(u) (x|\lambda)_{n-l} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.2)$$

Thus by (2.1) and (2.2), we get

$$h_{n,\lambda}(x|u) = \sum_{l=0}^n \binom{n}{l} h_{l,\lambda}(u) (x|\lambda)_{n-l}. \quad (2.3)$$

From (2.1), we can derive the following recurrence relation:

$$\begin{aligned} 1-u &= (1+\lambda t)^{\frac{1}{\lambda}} \sum_{n=0}^{\infty} h_{n,\lambda}(u) \frac{t^n}{n!} - u \sum_{n=0}^{\infty} h_{n,\lambda}(u) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} h_{n-l,\lambda}(u) (1|\lambda)_l \right) \frac{t^n}{n!} - u \sum_{n=0}^{\infty} h_{n,\lambda}(u) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} (h_{n,\lambda}(1|u) - u h_{n,\lambda}(u)) \frac{t^n}{n!}. \end{aligned} \quad (2.4)$$

By comparing of the coefficients on the both sides of (2.4), we get

$$h_{n,\lambda}(1|u) - u h_{n,\lambda}(u) = \begin{cases} 1-u, & \text{if } n=0 \\ 0, & \text{if } n>0. \end{cases} \quad (2.5)$$

Note that

$$\lim_{\lambda \rightarrow 0} \frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u} (1+\lambda t)^{\frac{x}{\lambda}} = \frac{1-u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(x|u) \frac{t^n}{n!}. \quad (2.6)$$

From (2.1) and (2.6), we have

$$H_n(x|u) = \lim_{\lambda \rightarrow 0} h_{n,\lambda}(x|u), \quad (n \geq 0). \quad (2.7)$$

We observe that

$$\begin{aligned} & \frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u} (1+\lambda t)^{\frac{x+1}{\lambda}} - \frac{u(1-u)}{(1+\lambda t)^{\frac{1}{\lambda}} - u} (1+\lambda t)^{\frac{x}{\lambda}} \\ &= (1-u)(1+\lambda t)^{\frac{x}{\lambda}} = (1-u) \sum_{n=0}^{\infty} (x|\lambda)_n \frac{t^n}{n!}. \end{aligned} \quad (2.8)$$

Therefore, by (2.1) and (2.8), we get

$$h_{n,\lambda}(x+1|u) - u h_{n,\lambda}(x|u) = (x|\lambda)_n (1-u), \quad (n \geq 0). \quad (2.9)$$

Therefore, we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$h_{n,\lambda}(x|u) = \sum_{l=0}^n \binom{n}{l} h_{l,\lambda}(u) (x|\lambda)_{n-l}$$

and

$$\frac{h_{n,\lambda}(x+1|u)}{1-u} - \frac{u}{1-u} h_{n,\lambda}(x|u) = (x|\lambda)_n.$$

In particular,

$$h_{n,\lambda}(1|u) - u h_{n,\lambda}(u) = (1-u) \delta_{0,n},$$

where $\delta_{n,k}$ is the Kronecker's symbol.

By (2.1), we get

$$\begin{aligned} \sum_{n=0}^{\infty} h_{n,-\lambda}(u) \frac{t^n}{n!} &= \frac{1-u}{(1-\lambda t)^{-\frac{1}{\lambda}} - u} = \frac{1-u}{1-u(1-\lambda t)^{\frac{1}{\lambda}}} (1-\lambda t)^{\frac{1}{\lambda}} \\ &= \frac{1-u^{-1}}{(1-\lambda t)^{\frac{1}{\lambda}} - u^{-1}} (1-\lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} h_{n,\lambda}(1|u^{-1}) (-1)^n \frac{t^n}{n!}. \end{aligned} \quad (2.10)$$

In general,

$$\begin{aligned} \sum_{n=0}^{\infty} h_{n,-\lambda}(-x|u) \frac{t^n}{n!} &= \frac{1-u}{(1-\lambda t)^{-\frac{1}{\lambda}} - u} (1-\lambda t)^{\frac{x}{\lambda}} = \frac{1-u^{-1}}{(1-\lambda t)^{\frac{1}{\lambda}} - u^{-1}} (1-\lambda t)^{\frac{x+1}{\lambda}} \\ &= \sum_{n=0}^{\infty} h_{n,\lambda}(x+1|u^{-1}) (-1)^n \frac{t^n}{n!}. \end{aligned} \quad (2.11)$$

Therefore, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$(-1)^n h_{n,-\lambda}(-x|u) = h_{n,\lambda}(x+1|u^{-1}).$$

In particular,

$$(-1)^n h_{n,-\lambda}(u) = (-1)^n h_{n,\lambda}(1|u^{-1}).$$

We observe that

$$\begin{aligned}
-\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}(1+\lambda t)^{\frac{x}{\lambda}} &= \frac{u^d(1-u^{-1})}{(1+\lambda t)^{\frac{d}{\lambda}}-u^d} \sum_{l=0}^{d-1} u^{-l}(1+\lambda t)^{\frac{l+x}{\lambda}} \\
&= \frac{u^d(1-u^{-1})}{1-u^d} \sum_{l=0}^{d-1} \frac{u^{-l}(1-u^d)}{(1+\lambda t)^{\frac{d}{\lambda}}-u^d} (1+\lambda t)^{\frac{l+x}{\lambda}} \\
&= \frac{(1-u^{-1})}{1-u^d} \sum_{l=0}^{d-1} \sum_{n=0}^{\infty} u^{d-l} h_{n, \frac{\lambda}{d}} \left(\frac{l+x}{d} \middle| u^d \right) d^n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left\{ \left(\frac{1-u^{-1}}{1-u^d} \right) d^n \sum_{a=0}^{d-1} u^{d-a} h_{n, \frac{\lambda}{d}} \left(\frac{a+x}{d} \middle| u^d \right) \right\} \frac{t^n}{n!}.
\end{aligned} \tag{2.12}$$

Therefore, by (2.1) and (2.12), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, $d \in \mathbb{N}$, we have*

$$h_{n, \lambda}(x|u) = d^n \left(\frac{-1+u^{-1}}{1-u^d} \right) \sum_{a=0}^{d-1} u^{d-a} h_{n, \frac{\lambda}{d}} \left(\frac{a+x}{d} \middle| u^d \right).$$

For $r \in \mathbb{N}$, let us consider the degenerate Frobenius-Euler polynomials of order r which are given by the generating function to be

$$\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u} \right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} h_{n, \lambda}^{(r)}(x|u) \frac{t^n}{n!}. \tag{2.13}$$

When $x = 0$, $h_{n, \lambda}^{(r)}(u) = h_{n, \lambda}^{(r)}(0|u)$ are called *degenerate Frobenius-Euler numbers of order r* .

From (2.13), we can easily derive the following equation.

$$h_{n, \lambda}^{(r)}(x) = \sum_{l=0}^n \binom{n}{l} h_{l, \lambda}^{(r)}(u)(x|\lambda)_{n-l}, \quad (n \geq 0). \tag{2.14}$$

By (2.13), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} h_{n, \lambda}^{(r)}(x+y|\lambda) \frac{t^n}{n!} &= \left(\sum_{l=0}^{\infty} h_{l, \lambda}^{(r)}(x) \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} (x|\lambda)_m \frac{t^m}{m!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n h_{l, \lambda}^{(r)}(x)(x|\lambda)_{n-l} \binom{n}{l} \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.15}$$

Therefore, by (2.15), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$h_{n, \lambda}^{(r)}(x+y|\lambda) = \sum_{l=0}^n \binom{n}{l} h_{l, \lambda}^{(r)}(x)(x|\lambda)_{n-l}.$$

From Theorem 2.4, we note that $h_{n, \lambda}^{(r)}(x|u)$ is a Sheffer sequence.

By replacing t by $\frac{1}{\lambda}(e^{\lambda t} - 1)$ in (2.13), we get

$$\begin{aligned} \left(\frac{1-u}{e^t - u}\right)^r e^{xt} &= \sum_{n=0}^{\infty} h_{n,\lambda}^{(r)}(x|u) \frac{1}{n!} \left(\frac{1}{\lambda}(e^{\lambda t} - 1)\right)^n \\ &= \sum_{n=0}^{\infty} h_{n,\lambda}^{(r)}(x|u) \lambda^{-n} \sum_{m=n}^{\infty} S_2(m, n) \lambda^m \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^m h_{n,\lambda}^{(r)}(x|u) \lambda^{m-n} S_2(m, n) \right) \frac{t^m}{m!}. \end{aligned} \quad (2.16)$$

As is well known, the higher-order Frobenius-Euler polynomials are defined by the generating function to be

$$\left(\frac{1-u}{e^t - u}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|u) \frac{t^n}{n!}. \quad (2.17)$$

Thus, by (2.16) and (2.17), we get

$$H_n^{(r)}(x|u) = \sum_{n=0}^m h_{n,\lambda}^{(r)}(x|u) \lambda^{m-n} S_2(m, n). \quad (2.18)$$

Therefore, by (2.18), we obtain the following theorem.

Theorem 2.5. *For $m \geq 0$, we have*

$$H_n^{(r)}(x|u) = \sum_{n=0}^m h_{n,\lambda}^{(r)}(x|u) \lambda^{m-n} S_2(m, n).$$

By replace t by $\log(1 + \lambda t)^{\frac{1}{\lambda}}$ in (2.17), we have

$$\begin{aligned} \left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} &= \sum_{n=0}^{\infty} H_n^{(r)}(x|u) \frac{1}{n!} \frac{1}{\lambda^n} (\log(1 + \lambda t))^n \\ &= \sum_{n=0}^{\infty} H_n^{(r)}(x|u) \lambda^{-n} \sum_{m=n}^{\infty} S_1(m, n) \lambda^m \frac{t^m}{n!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^m H_n^{(r)}(x|u) \lambda^{m-n} S_1(m, n) \right) \frac{t^m}{m!}. \end{aligned} \quad (2.19)$$

Therefore, by (2.13) and (2.18), we obtain the following theorem.

Theorem 2.6. *For $m \geq 0$, we have*

$$h_{n,\lambda}^{(r)}(x|u) = \sum_{n=0}^m H_n^{(r)}(x|u) \lambda^{m-n} S_1(m, n).$$

We observe that

$$\begin{aligned} &\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u}\right)^r (1+\lambda t)^{\frac{x+1}{\lambda}} - u \left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} \\ &= \left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u}\right)^{r-1} (1+\lambda t)^{\frac{x}{\lambda}} (1-u) = \sum_{n=0}^{\infty} h_{n,\lambda}^{(r-1)}(x|u) (1-u) \frac{t^n}{n!}. \end{aligned} \quad (2.20)$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$\frac{1}{1-u} \left\{ h_{n,\lambda}^{(r)}(x+1|u) - u h_{n,\lambda}^{(r)}(x|u) \right\} = h_{n,\lambda}^{(r-1)}(x|u).$$

Remark. Let $u = -1$. Then, by (2.1), we get

$$\frac{1}{t} \frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} h_{n,\lambda}(x| -1) \frac{t^n}{n!}. \quad (2.21)$$

Thus, by (2.21), we get

$$\frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} (1+\lambda t)^{\frac{x}{\lambda}} = t \sum_{n=0}^{\infty} h_{n,\lambda}(x| -1) \frac{t^n}{n!}.$$

Now, we define the degenerate Genocchi polynomials which are given by the generating function to be

$$\frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} g_{n,\lambda}(x) \frac{t^n}{n!}. \quad (2.22)$$

From (2.21) and (2.22), we have

$$g_{0,\lambda}(x) = 0 \text{ and } h_{n,\lambda}(x| -1) = \frac{1}{n+1} g_{n+1,\lambda}(x), \quad (n \geq 0).$$

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